

NON-DIVERGENCE PARABOLIC EQUATIONS OF SECOND ORDER WITH CRITICAL DRIFT IN MORREY SPACES

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ABSTRACT. We consider uniformly parabolic equations and inequalities of second order in the non-divergence form with drift

$$-u_t + Lu = -u_t + \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u = 0 (\geq 0, \leq 0)$$

in some domain $\Omega \subset \mathbb{R}^{n+1}$. We prove a variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate with L^p norm of the inhomogeneous term for some number $p < n+1$. Based on it, we derive the growth theorems and the interior Harnack inequality. In this paper, we will only assume the drift b is in certain Morrey spaces defined below which are critical under the parabolic scaling but not necessarily to be bounded. This is a continuation of the work in [GC].

1. INTRODUCTION

1.1. General Introduction. The qualitative properties of solutions to partial differential equations have been intensively studied for a long time. Following [GC], in this note, we continue our discussion on the qualitative properties of solutions to the uniform parabolic equation of non-divergence form with drift,

$$(1.1) \quad -u_t + Lu := -u_t + \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u = 0$$

and the associated inequalities: $-u_t + Lu \geq 0$ and $-u_t + Lu \leq 0$. Throughout the paper, we use the notations $D_i := \frac{\partial}{\partial x_i}$, $D_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$ and $u_t := \frac{\partial u}{\partial t}$. We assume $b = (b_1, \dots, b_n)$ and a_{ij} 's are real measurable, a_{ij} 's also satisfy the *uniform parabolicity condition*

$$(1.2) \quad \forall \xi \in \mathbb{R}^n, \nu^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(X) \xi_i \xi_j, \quad \sum_{i,j=1}^n a_{ij}^2 \leq \nu^2$$

with some constant $\nu \geq 1$, $\forall X = (x, t)$ in the domain of definition $\Omega \subset \mathbb{R}^{n+1}$.

For the drift b , we will only require it is in certain Morrey spaces which are critical under the parabolic scaling. To formulate our setting more precisely, we define Morrey spaces as following: given some constants $p, q \geq 1$ and $\alpha \geq 0$ satisfying

$$(1.3) \quad \frac{n}{p} + \frac{2}{q} - \alpha = 1,$$

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on the domain of definition Ω , we define

$$(1.4) \quad M_{p,q}^\alpha(\Omega) := \left\{ f \in L_x^p L_t^q(\Omega); \|f\|_{M_{p,q}^\alpha(\Omega)} := \sup_{Q_r \subset \Omega, r>0} r^{-\alpha} \|f\|_{L_x^p L_t^q(Q_r)} < \infty \right\}$$

where Q_r is the standard parabolic cylinder defined in Definition 4 and

$$\|f\|_{L_x^p L_t^q} := \left(\int \left[\int |f(x,t)|^q dt \right]^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

We will focus on a particular case with $p = q = \frac{1}{\alpha} = n + 1$,

$$(1.5) \quad b \in M_{n+1,n+1}^{\frac{1}{n+1}}(\Omega)$$

with

$$(1.6) \quad \sup_{Q_r \subset \Omega, r>0} \left[\frac{1}{r} \int_{Q_r} |b|^{n+1} dx dt \right] =: S(\Omega) < \infty.$$

By "critical", we mean that with the $M_{n+1}^{\frac{1}{n+1}}(\Omega)$ norm, the drift is scaling invariant under the parabolic scaling: for $k > 0$,

$$x \rightarrow k^{-1}x, \quad t \rightarrow k^{-2}t.$$

Indeed, suppose u satisfies

$$-u_t + \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u = 0.$$

in a domain $Q \in \mathbb{R}^{n+1}$. Then for any constant $k > 0$, let

$$\tilde{x} = k^{-1}x, \quad \tilde{t} = k^{-2}t.$$

Then $\tilde{u}(\tilde{x}, \tilde{t}) = u(r\tilde{x}, r^2\tilde{t})$ satisfies the equation

$$-\tilde{u}_{\tilde{t}} + \sum_{ij} \tilde{a}_{ij} D_{ij} \tilde{u} + \sum_i \tilde{b}_i D_i \tilde{u} = 0,$$

in $Q_k := \{(x, t), (kx, k^2t) \in Q\}$. Note that $\tilde{b} = kb$, so

$$S^{\frac{1}{n+1}}(\Omega_k) = \left\| \tilde{b} \right\|_{M_{n+1,n+1}^{\frac{1}{n+1}}(\Omega_k)} = \|b\|_{M_{n+1,n+1}^{\frac{1}{n+1}}(\Omega)} = S^{\frac{1}{n+1}}(\Omega).$$

In general, regarding the scaling, intuitively, there is a competition between the transport term and the diffusion part. One might expect that for the supercritical scaling case, $\frac{n}{p} + \frac{2}{q} - \alpha > 1$: the solutions of the equations have discontinuities [SVZ]. For the critical situation we are considering here, we have Hölder continuous solutions, see Theorem 22. Finally, if the drift is subcritical with respect to the scaling, i.e. $\frac{n}{p} + \frac{2}{q} - \alpha < 1$, we expect the solutions will be smooth. We should notice $L_{x,t}^{n+1}$ ($p = q = n + 1, \alpha = 0$) is supercritical with respect to the parabolic scaling. We will discuss a concrete example in the appendix.

We will concentrate on the growth theorems and the interior Harnack inequality for parabolic equations in non-divergence form with critical drift. In order to derive them, we prove a variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate, Theorem 2. This variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate enable us to estimate the supremum of a solution to $-u_t + Lu = f$ in a bounded

Lipschitz domain in \mathbb{R}^{n+1} in terms of the Dirichlet data on the boundary and the L^p norm of f with some constant $p < n + 1$ depending on n, ν, S .

With the assumptions and preparations above, the main results in this paper are then expressed by Theorems 2 and 3.

Definition 1. Given $p \geq 1$, for any open set $\Omega \subset \mathbb{R}^{n+1}$, we define the space

$$(1.7) \quad W_p(\Omega) := C(\Omega) \cap W_{p,p}^{2,1}(\Omega),$$

where $f \in W_{p,p}^{2,1}(\Omega)$ means $f_t, D_i f, D_{ij} f \in (L_x^p L_t^p)_{loc}$.

Theorem 2. *Under the assumptions above, there are constants $p := p(\nu, n, S) < n + 1$ and N depending on ν, n and S such that if $u \in C(\overline{\Omega}) \cap W_p(\Omega)$ satisfies $-u_t + Lu \geq f$ in Ω with $f \in L^p(\Omega)$, and $u \leq 0$ on $\partial_p \Omega$, then*

$$(1.8) \quad \sup_{\Omega} u \leq N r^{2 - \frac{n+2}{p}} \|f\|_{L^p}$$

where r is the diameter of Ω .

With the help of the variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate (1.8), we can obtain:

Theorem 3 (Interior Harnack Inequality). *Suppose $u \in C(\overline{Q_{2r}(Y)}) \cap W_p(Q_{2r}(Y))$ and $-u_t + Lu = 0$ in $Q_{2r}(Y)$, $Y = (y, s) \in \mathbb{R}^{n+1}$ and $r > 0$. If $u \geq 0$, then*

$$(1.9) \quad \sup_{Q^0} u \leq N \inf_{Q_r} u,$$

where $N = N(n, \nu, S)$ and $Q^0 = B_r(y) \times (s - 3r^2, s - 2r^2)$.

Harnack inequalities have many important applications, not only in differential equations, but also in other areas, such as diffusion processes, geometry, etc. Unlike the classical maximum principle, the interior Harnack inequality is far from obvious. For elliptic and parabolic equations with measurable coefficients in the divergence form, it was proved by Moser in the papers [M61], [M64]. However, a similar result for non-divergence equations was obtained 15 years later after Moser's papers by Krylov and Safonov [KS], [S80] in 1978-80. Their proofs relied on some improved versions of growth theorems from the book by Landis [EML]. These growth theorems control the behavior of (sub-, super-) solutions of second order elliptic and parabolic equations in terms of the Lebesgue measure of areas in which solutions are positive or negative. So certainly, if some estimate can directly import the information about measure, it should be useful. In [FS], Ferretti and Safonov used growth theorems as a common background for both divergence and non-divergence equations and used these three growth theorems to derive the interior Harnack inequality. Even in the one-dimensional case, the Harnack inequality fails for equations of a "joint" structure, which combine both divergence and non-divergence parts. One can find detailed discussion in [CS13].

At the beginning, the interior Harnack inequality was proved with bounded drift. Later on, this condition was relaxed to subcritical drift b . For the subcritical case, we can always rescale the problem. In small scale, the drift will work like a perturbation from the case without drift. But for the critical situation, our common tricks do not work. One can find a historical overview of this progress in [NU]. For non-divergence elliptic equations of second order, in [S10], Safonov shown the interior Harnack inequality for the scaling critical case $b \in L^n$. In [GC],

the author proved the interior Harnack inequality for parabolic equations of second order in non-divergence form with the drift b in certain Lebesgue spaces which are scaling invariant. In this note which appears as a companion of the earlier paper [GC], we consider the case when the drift $b \in M_{n+1, n+1}^{\frac{1}{n+1}}(\Omega)$ which is again scaling invariant. Similar results for both divergence form elliptic and parabolic equations are presented in [NU].

We will follow the unified approach to growth theorems and the interior Harnack inequality developed in [FS]. For this purpose, we need to prove three growth theorems and derive the interior Harnack inequality as a consequence for parabolic equations with critical drift formulated as above. In order to take the measure conditions into account and help us carry out growth theorems, we discuss the variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate (1.8). Although we only consider the case $b \in M_{n+1, n+1}^{\frac{1}{n+1}}(\Omega)$, one can see from the proofs, our approach works well for other pairs (p, q, α) satisfying condition (1.3) with $\alpha > 0$ provided the associated standard Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate holds, see Sections 2 and 3. For the sake of simplicity, we assume that all functions (coefficients and solutions) are smooth enough. It is easy to get rid of extra smoothness assumptions by means of standard approximation procedures, see Section 7. We should notice that it is important to have appropriate estimates for solutions with constants depending only on the prescribed quantities, such as the dimension n , the parabolicity constant, etc., but not depending on “additional” smoothness.

This paper is organized as follows: In Section 1, we introduce our basic assumptions and notations. In Section 2, we formulate a weak version of the classical maximum principle, the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate, and some consequences of it. In Section 3, we establish a variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso based on some estimates of Green’s function. In Sections 4, 5, 6, we formulate and prove three growth theorems and prove the interior Harnack inequality. Finally, in Section 7, we use approximation to show all results are valid without smoothness assumption. In the appendix, an example of loss of continuity of the solution to a parabolic equation with drift $b \in L_{x,t}^{n+1}$ will be presented.

1.2. Notations: In this paper, we use summation convention.

“ $A := B$ ” or “ $B =: A$ ” is the definition of A by means of the expression B .

\mathbb{R}^n is the n -dimensional Euclidean space, $n \geq 1$, with points $x = (x_1, \dots, x_n)^t$, where x_i ’s are real numbers. Here the symbol t stands for the transposition of vectors which indicates that vectors in \mathbb{R}^n are treated as column vectors. For $x = (x_1, \dots, x_n)^t$ and $y = (y_1, \dots, y_n)^t$ in \mathbb{R}^n , the scalar product $(x, y) := \sum x_i y_i$, the length of x is $|x| := (x, x)^{\frac{1}{2}}$.

For a Borel set $\Gamma \subset \mathbb{R}^n$, $\bar{\Gamma} := \Gamma \cup \partial\Gamma$ is the closure of Γ , $|\Gamma|$ is the n -dimensional Lebesgue measure of Γ . Sometimes we use the same notation for the surface measure of a subset Γ of a smooth surface S .

For real numbers c , we denote $c_+ := \max(c, 0)$, $c_- := \max(-c, 0)$.

In order to formulate our results, we need some standard definitions and notations for the setting of parabolic equations.

Definition 4. Let Q be an open connected set in \mathbb{R}^{n+1} , $n \geq 1$. The parabolic boundary $\partial_p Q$ of Q is the set of all points $X_0 = (x_0, t_0) \in \partial Q$, such that there exists a continuous function $x = x(t)$ on the interval $[t_0, t_0 + \delta)$ with values in \mathbb{R}^n , such that $x(t_0) = x_0$ and $(x(t), t) \in Q$ for all $t \in (t_0, t_0 + \delta)$. Here $x = x(t)$ and

$\delta > 0$ depend on X_0 . In particular, for cylinders $Q_U = U \times (0, T)$ with $U \subset \mathbb{R}^n$, the parabolic boundary $\partial_p Q_U := (\partial_x Q_U) \cup (\partial_t Q_U)$, where $\partial_x Q_U := (\partial U) \times (0, T)$, $\partial_t Q_U := U \times \{0\}$.

We will use the following notation for the "standard" parabolic cylinder. For $Y = (y, s)$ and $r > 0$, we define $Q_r(Y) := B_r(y) \times (s - r^2, s)$, where $B_r(y) := \{x \in \mathbb{R}^n : |x - y| < r\}$.

2. PRELIMINARIES

In this section, we briefly discuss some well-known theorems and results which are crucial for us to carry out the discussion in the later parts of this paper. We use the notation $u \in W_{n+1}$ in the sense of Definition 1.

Theorem 5 (Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate). *Suppose $u \in W_{n+1}(\Omega)$, $\Omega \subset Q_r$ and $-u_t + Lu \geq f$. If $\sup_{\partial_p \Omega} u \leq 0$, then*

$$(2.1) \quad \sup_{\Omega} u \leq N \left(r^{\frac{n}{n+1}} + \|b\|_{L^{n+1}}^n \right) \|f\|_{L^{n+1}}$$

where $N = N(n, \nu)$.

One can find the detailed proof of the above standard version of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate in [AIN] and [GL2]. From (2.1), one can see if we take $r^{\frac{n}{n+1}}$ out of the bracket, we will have

$$\sup_{\Omega} u \leq \sup_{\partial \Omega} u + Nr^{\frac{n}{n+1}} \left(1 + \|b\|_{M_{n+1}^{\frac{n}{n+1}}(\Omega)}^n \right) \|f\|_{L^{n+1}}.$$

So it is natural to consider the case $b \in M_{n+1}^{\frac{n}{n+1}}(\Omega)$.

Remark 6. In [AIN], Nazarov shown the Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate holds for the drift $b \in L_x^p L_t^q$, i.e.,

$$\|b\|_{L_x^p L_t^q} = \left(\int \left[\int |b(x, t)|^q dt \right]^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty,$$

for

$$\frac{n}{p} + \frac{2}{q} \leq 1, \quad p, q \geq 1.$$

The proof was based on Krylov's ideas and methods [NVK]. We believe if b is in other scaling invariant Morrey space, and the Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate holds for the corresponding case, then the proofs in this note also hold.

Theorem 7 (Maximal Principle). *Let Q be a bounded open set in \mathbb{R}^{n+1} , and let a function $u \in C^{2,1}(\bar{Q} \setminus \partial_p Q) \cap C(\bar{Q})$ satisfy the inequality $-u_t + Lu \geq 0$ in Q . Then*

$$(2.2) \quad \sup_Q u = \sup_{\partial_p Q} u$$

As an easy consequence of the maximal principle and the Alexandrov-Bakelman-Pucci-Krylov-Tso estimate, we have the well-known comparison principle.

Theorem 8 (Comparison Principle). *Let Q be a bounded domain in \mathbb{R}^{n+1} , $u, v \in C^{2,1}(\bar{Q} \setminus \partial_p Q) \cap C(\bar{Q})$, $-u_t + Lu \leq -v_t + Lv$ in Q , and $u \geq v$ on $\partial_p Q$, then $u \geq v$ on \bar{Q} .*

3. A VARIANT OF ALEKSANDROV-BAKELMAN-PUCCI-KRYLOV-TSO ESTIMATE

3.1. Estimates of Green's function: In this subsection, we show some estimates for Green's function following [FSt, GL] in order to show the variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate (1.8).

We consider the non-divergence second order parabolic equations of the form

$$(3.1) \quad -u_t + Lu = -u_t + \sum_{i,j=1}^n a_{ij} D_{ij}u + \sum_{i=1}^n b_i D_i u$$

defined on some Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$.

We define Green's function $G : \Omega \times \Omega \rightarrow \mathbb{R}$ satisfies following properties: if $u(x, t)$ has the form

$$(3.2) \quad u(x, t) = \int_{\Omega} G(x, t, y, s) f(y, s) dy ds$$

then it solves the Dirichlet problem:

$$-u_t + Lu = -f,$$

and

$$u = 0$$

on $\partial_p \Omega$. Throughout this subsection, we will assume all coefficients are smooth, then the existence of Green's function is guaranteed. We will adapt the ideas in [GL] to our parabolic setting. The key step is to verify Lemma 2.1 in [GL] holds in parabolic case with $b \in M_{n+1}^{\frac{1}{n+1}}(\Omega)$. We formulate the following lemma which is similar to Lemma 2.1 in [GL] under condition (1.6).

Lemma 9. *Let $(x_1, t_1) \in \Omega$, then we can choose $1 > \rho > 0$ depending on ν, n and S such that $Q_{2\rho}(x_1, t_1) \subset \Omega$ and there is a constant C depends on ν, n and S , we have*

$$(3.3) \quad \left(\int_{Q_{\rho}(x_1, t_1)} G(x, t, y, s)^{\frac{n+1}{n}} dy ds \right)^{\frac{n}{n+1}} \leq \frac{C}{\rho^{\frac{n+2}{n+1}}} \int_{Q_{\frac{\rho}{2}}(x_1, t_1)} G(x, t, y, s) dy ds$$

for any $(x, t) \in \Omega$ and $t \leq t_1$ where G is Green's function defined as (3.2).

Proof. Without loss of generality, we may assume $(x_1, t_1) = (0, 0)$. We will also use Q_r to denote $Q_r(0, 0)$. Clearly, it will be sufficient to show that

$$(3.4) \quad \int_{Q_{\rho}} G(x, t, y, s) f(y, s) dy ds \leq \frac{C}{\rho^{\frac{n+2}{n+1}}} \int_{Q_{\frac{\rho}{2}}} G(x, t, y, s) dy ds$$

for any non-negative function $f \in L^{n+1}(Q_{\rho})$ with $\int_{Q_{\rho}} f^{n+1} = 1$. We fix such a f , and then construct u_1 and u_2 as following:

$$(3.5) \quad u_1(x, t) = \int_{Q_\rho} G(x, t, y, s) f(y, s) dy ds,$$

$$(3.6) \quad u_2(x, t) = \int_{Q_{\frac{\rho}{2}}} G(x, t, y, s) dy ds.$$

We define

$$(3.7) \quad \eta(x, t) := 1 - \frac{1}{4\rho^2} \|x\|^2 - \frac{1}{4\rho^2} t.$$

There are positive constants N_1 , δ and q (determined by the parabolicity ν and the dimension n), such that

$$(3.8) \quad -D_t \eta^q + \sum_{ij} a_{ij} D_{ij} \eta^q \geq \begin{cases} 0 & Q_{2\rho} \setminus Q_{\frac{\rho}{2}}, \\ -N_1 \rho^{-2} & Q_{\frac{\rho}{2}}, \end{cases}$$

and

$$(3.9) \quad \eta^q > \delta$$

in Q_ρ .

Let u be the solution of the Dirichlet problem

$$(3.10) \quad -u_t + Lu = -f$$

in $Q_{2\rho}$ and

$$(3.11) \quad u = 0$$

on $\partial_p Q_{2\rho}$.

By the Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate we have

$$(3.12) \quad u \leq NN_2(\rho)^{\frac{n}{n+1}} = k_1 \rho^{\frac{n}{n+1}},$$

where k_1 depends on ν , S and n .

We set $C = \frac{2N_1NN_2}{\delta}$,

$$(3.13) \quad w := u_1 - \frac{C}{\rho^{\frac{n+2}{n+1}}} u_2,$$

$$(3.14) \quad \bar{w} := w - u + \frac{2NN_2\rho^{\frac{n}{n+1}}}{\delta} \eta^q$$

and

$$(3.15) \quad M := \max \left\{ 0, \sup_{\partial_p Q_{2\rho}} w \right\}.$$

Then by some computations, we know that

$$(3.16) \quad -\bar{w}_t + L\bar{w} \geq N_3|b|$$

in $Q_{2\rho}$ where N_3 depends on ν , S and n . And $\bar{w} \leq M$ on the parabolic boundary of $Q_{2\rho}$. By the Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate again, we have

$$(3.17) \quad \bar{w} \leq NN_2\rho S^{\frac{1}{n+1}} + M$$

in $Q_{2\rho}$. In Q_ρ with $\eta^q > \delta$, we obtain

$$(3.18) \quad w \leq NN_2\rho S^{\frac{1}{n+1}} + M + NN_2(\rho)^{\frac{n}{n+1}} - \frac{2NN_2\rho^{\frac{n}{n+1}}}{\delta}\eta^q \leq M$$

when we take ρ small enough since $\rho < \rho^{\frac{n}{n+1}}$ when $\rho < 1$. Here the smallness condition only depends on prescribed constants.

By our construction, it is clear that

$$(3.19) \quad -w_t + Lw = 0$$

in $\Omega \setminus Q_\rho$.

$$(3.20) \quad w \leq M$$

in $\overline{Q_\rho}$ and the parabolic boundary of Ω . By the maximal principle, we have $M = 0$. So $w \leq 0$ in all $(x, t) \in \Omega$ with $t \leq 0$. Therefore, after we decipher w , we get for ρ small enough (here the smallness condition only depends on ν , S and n), we conclude that

$$\int_{Q_\rho} G(x, t, y, s) f(y, s) dy ds \leq \frac{C}{\rho^{\frac{n+2}{n+1}}} \int_{Q_{\frac{\rho}{2}}} G(x, t, y, s) dy ds,$$

which implies

$$\left(\int_{Q_\rho(x_1, t_1)} G(x, t, y, s)^{\frac{n+1}{n}} dy ds \right)^{\frac{n}{n+1}} \leq \frac{C}{\rho^{\frac{n+2}{n+1}}} \int_{Q_{\frac{\rho}{2}}(x_1, t_1)} G(x, t, y, s) dy ds.$$

□

With Lemma 9, we can proceed to the prove of the variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate (1.8) similar to results in [GL] and [FSt]. Since every quantity we are considering here is scaling invariant, we may rescale our setting to a domain with diameter 1. So it suffices estimate the integrability of Green's function in a domain with diameter 1.

Theorem 10. *Under the same assumptions as above, then there are constants $q := q((\nu, n, S) > \frac{n+1}{n})$ and C only depending on ν , n and S such that*

$$(3.21) \quad \left(\int_{\Omega} G(x, t, y, s)^q dy ds \right)^{\frac{1}{q}} \leq C$$

where G is Green's function and the diameter of Ω is 1.

Proof. By our assumptions Ω is bounded. Suppose

$$(3.22) \quad \Omega' = \left\{ X := (x, t) \in \Omega, d_\Omega(X) < \frac{1}{2} \right\}$$

where

$$d_\Omega(X) := \sup \{ \rho > 0 : Q_\rho(X) \subset \Omega \}.$$

For arbitrary $(x', t') \in \Omega'$ and let $r < \frac{1}{2}$, then $Q_r(x', t') \subset \Omega$. For each $Q_r(x', t')$, we can use finite many $Q_\rho(x_i, t_i)$ to cover it, where ρ is chosen small enough so that

the conditions of Lemma 9 are satisfied. It is clear that the number of $Q_\rho(x_i, t_i)$ can be bounded by a positive constant C_1 depending on ν , n and S and the diameter of Ω . Let $\rho = C_2 r$, then we have

$$(3.23) \quad \left(\int_{Q_r(x', t')} G(x, t, y, s)^{\frac{n+1}{n}} dy ds \right)^{\frac{n}{n+1}} \leq \frac{C_3}{\rho^{\frac{n+2}{n+1}}} \int_{Q_r(x', t')} G(x, t, y, s) dy ds$$

where C_3 depends on C_1 , C_2 and C in Lemma 9.

We rewrite then inequality (3.23) as

$$(3.24) \quad \left(\int_{Q_r(x', t')} G(x, t, y, s)^{\frac{n+1}{n}} dy ds \right)^{\frac{n}{n+1}} \leq C_3 \int_{Q_r(x', t')} G(x, t, y, s) dy ds$$

By Gehring's lemma [MM], we get

$$(3.25) \quad \left(\int_{\Omega'} G(x, t, y, s)^q dy ds \right)^{\frac{1}{q}} \leq C_4 \int_{\Omega'} G(x, t, y, s) dy ds$$

for some

$$q > \frac{n+1}{n},$$

where C_4 depends on C_1 , C_2 and C in Lemma 9.

$$(3.26) \quad \int_{\Omega'} G(x, t, y, s) dy ds < C_5$$

where the constant C_5 depends on ν , n and S by the standard Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate, Theorem 5. Hence there is a constant C' such that

$$(3.27) \quad \left(\int_{\Omega'} G(x, t, y, s)^q dy ds \right)^{\frac{1}{q}} \leq C'$$

where C' only depends on ν , n and S . In order to get the same result for Ω , we can extend the all coefficients to a domain $\tilde{\Omega}$ with comparable quantities, such that $\Omega \subset \tilde{\Omega}$ and $\forall X := (x, t) \in \Omega$, $d_{\tilde{\Omega}}(X) < 1$. Then Green's function \tilde{G} for $\tilde{\Omega}$ will satisfy the same result as (3.27). And we know $\tilde{G} \geq G$ by the maximal principle and the comparison principle. Hence we have

$$(3.28) \quad \left(\int_{\Omega} G(x, t, y, s)^q dy ds \right)^{\frac{1}{q}} \leq C$$

where C depends on ν , n and S . □

Remark 11. We can see from the proof, we just need $b \in M_{n+1}^\alpha(\Omega)$ for α positive, the above arguments work too provided the associated Aleksandrov-Bakelman-Pucci-Krylov-Tso Estimate holds. The point we choose $\alpha = \frac{1}{n+1}$ is that the space is scaling invariant under the parabolic scaling.

3.2. A variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso Estimate: In this subsection, we still keep all the assumptions above. We will obtain a variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate for a solution of the second order parabolic equation differential equation $-u_t + Lu = f$ in a bounded Lipschitz domain in \mathbb{R}^{n+1} with Dirichlet problem condition. We estimate the maximal of u in terms of the Dirichlet data (boundary value) on the boundary and the L^p norm of f with $p < n+1$. For elliptic and parabolic equations without drift, one can find some references in [CS, EML, GL, FSt].

Theorem. *Under the same assumptions above, define $p = s \frac{q}{q-1}$, where $q > \frac{n+1}{n}$ is the constant from Theorem 10. Suppose $u \in C(\bar{\Omega}) \cap W_p(\Omega)$ satisfies $-u_t + Lu \geq f$ in Ω and $u \leq 0$ on $\partial_p \Omega$. Then there is a constant N depends on ν , n and S such that*

$$(3.29) \quad \sup_{\Omega} u \leq N r^{2 - \frac{n+2}{p}} \|f\|_{L^p}$$

where r is the diameter of Ω .

Proof. Without loss of generality, we may assume the diameter of Ω is 1. We first prove the result holds in case 1: when u is smooth and the coefficients of the equations are smooth (Hölder). Then we use approximation to show in case 2: under general condition, the estimate (3.29) holds.

Case 1: we can represent u using:

$$(3.30) \quad u(x, t) = \int_{\Omega} G(x, t, y, s) [u_t - Lu] dy ds + \int_{\partial_p \Omega} \tilde{G}(x, t, y, s) u dy ds$$

where G is Green's function and \tilde{G} is from the Riesz representation theorem, since Ω is a Lipschitz domain. It is clear both of G and \tilde{G} are non-negative, hence under our assumptions we have

$$(3.31) \quad u(x, t) \leq \int_{\Omega} G(x, t, y, s) |f| dy ds$$

then by Theorem 10 and Hölder inequality, it is clear that

$$(3.32) \quad \sup_{\Omega} u \leq N \|f\|_{L^p},$$

where N depends on ν , n and S .

Case 2: First we assume $u \in C(\bar{\Omega}) \cap W_p^{2,1}(\Omega)$ and other functions are still smooth. Under these conditions, we pick up a sequence u^i which is smooth such that $u^i \rightarrow u$. It is clear there is a sequence $\{f_i\}_i^\infty$ such that $-(u_i)_t + Lu_i \geq f_i$ and $f_i \rightarrow f$ in $L^p(\Omega)$. So it is clear that

$$(3.33) \quad \sup_{\Omega} u \leq N \|f\|_{L^p}.$$

Finally, in general situation. Suppose $a_{ij}^k \rightarrow a_{ij}$ and $b_i^k \rightarrow b_i$ almost everywhere as $k \rightarrow \infty$. Define

$$(3.34) \quad L_k u = \sum_{ij} a_{ij}^k D_{ij} u + \sum_i b_i^k D_i u,$$

and

$$(3.35) \quad f_k = f + (L - L_k)u.$$

It is clear that $f_k \rightarrow f$ in $L^p(\Omega)$. Therefore

$$(3.36) \quad \sup_{\Omega} u \leq N \|f\|_{L^p}$$

holds.

Finally, after rescaling, we obtain (3.29) in the most general setting,

$$\sup_{\Omega} u \leq N r^{2 - \frac{n+2}{p}} \|f\|_{L^p}.$$

□

4. FIRST GROWTH THEOREM

Suppose R is the region in a cylinder where a subsolution u of our equation is positive. The first growth theorem, Theorem 12, basically tells us if the measure of R is small, then the maximal value of u over half of the cylinder is strictly less than the maximal value over the whole cylinder. In other words, it gives us some quantitative decay properties. The variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate (1.8) enables us to import information about the measure into our estimates.

Theorem 12 (First Growth Theorem). *Let a function $u \in C^{2,1}(\overline{Q_r})$ where $r > 0$ and $Q_r = Q_r(Y)$, in \mathbb{R}^{n+1} containing $Y := (y, s)$. Suppose $-u_t + Lu \geq 0$ in Q_r , then $\forall \beta_1 \in (0, 1)$, there exists $0 < \mu < 1$ such that if we know*

$$(4.1) \quad |\{u > 0\} \cap Q_r(Y)| \leq \mu |Q_r(Y)|,$$

then

$$(4.2) \quad \mathcal{M}_{\frac{r}{2}}(Y) \leq \beta_1 \mathcal{M}_r(Y),$$

where

$$\mathcal{M}_r(Y) := \max_{Q_r(Y)} u_+$$

We also notice that $\beta_1 \rightarrow 0^+$ as $\mu \rightarrow 0^+$.

Remark 13. First of all, we make some reductions. In our problem, we want to show under some conditions, given $-u_t + Lu \geq 0$ in a cylinder $Q_r(Y)$, and some information about the set $\{u \leq 0\}$, we want to show that

$$\mathcal{M}_{\frac{r}{2}}(Y) \leq \beta_1 \mathcal{M}_r(Y).$$

Clearly, in order to derive the above estimate, we only need to consider positive part of u . We observe that to obtain the above estimate, it actually suffices to show

$$(4.3) \quad u(Y) \leq \beta_1 \mathcal{M}_r(Y),$$

for some $\beta_1 \in (0, 1)$. Indeed, for an arbitrary point $Z \in Q_{\frac{r}{2}}(Y)$, we notice $Q_{\frac{r}{2}}(Z) \subset Q_r(Y)$, we can apply the above estimate (4.3) to $Q_{\frac{r}{2}}(Z)$ with Y replaced by Z and

r replaced by $\frac{r}{2}$ with some measure condition μ' . In consistent with the measure condition in the first growth theorem, we also observe that

$$|\{u > 0\} \cap Q_{\frac{r}{2}}(Z)| \leq |\{u > 0\} \cap Q_r(Y)| \leq \mu |Q_r| = 2^{n+2} \mu |Q_{\frac{r}{2}}(Z)|.$$

So we just need to take $\mu = 2^{-n-2} \mu'$ for the measure condition in the first growth theorem.

Proof. Since every quantity is scaling invariant, we might assume $r = 1$. And we can multiply u by a constant, so without loss of generality, we can also assume $\mathcal{M}_1(Y) = 1$. Also we assume $u(Y) > 0$, otherwise the result is trivial.

$$(4.4) \quad v(X) = v(x, t) = u(x, t) + t - s - |x - y|^2$$

in $Q := \{v > 0\} \cap Q_1(Y)$. Clearly, $Q \neq \emptyset$ since $v(Y) = u(Y) > 0$ and $Y \in \partial Q_1(Y)$. It is easy to see that $v \leq u$ in Q . By the measure condition, we have

$$|Q| \leq |\{u > 0\} \cap Q_1(Y)| \leq \mu_1 |Q_1(Y)| \leq \mu_1.$$

Note that $v \leq 0$ on $\partial_p Q_1(Y)$, so $v = 0$ on $\partial_p Q$. Since $-u_t + Lu \geq 0$, we know that

$$(4.5) \quad (-\partial_t + L)v \geq 0 - 1 - 2\text{trace}(a_{ij}) - 2|b| \geq -1 - 2nv^{-1} - 2|b|.$$

By the variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate (1.8) with some constant $p < n + 1$ and Hölder inequality,

$$(4.6) \quad u(Y) \leq N(\nu, n, S) \|-1 - 2nv^{-1} - 2|b|\|_{L^p(Q)} \leq N_1(\nu, n, S) \left(\mu^{\frac{1}{p}} + S^{\frac{1}{n+1}} \mu^{\frac{1}{p} - \frac{1}{n+1}} \right).$$

Now we can pick μ small enough so that for fixed β_1 , then we have

$$(4.7) \quad u(Y) < \beta_1.$$

It is also clear from the construction, $\beta_1 \rightarrow 0^+$ as $\mu \rightarrow 0^+$. \square

With the first growth theorem, we can do the following useful argument which is helpful for us to find a non-degenerate point to build a bridge between two regions we are interested in. Without loss of generality, we still assume $r = 1$, for $X \in Q_1(Y)$, we define

$$(4.8) \quad d(X) := \sup \{ \rho > 0 : Q_\rho(X) \subset Q_1(Y) \}.$$

Roughly here d plays roles of weights with which we can make sure the point we are interested in is not degenerate, i.e., it is in the interior of the cylinder. For $\gamma > 0$, we consider $d^\gamma u(x)$ instead of $u(x)$. $d^\gamma u(x)$ is a continuous function in $\overline{Q_1(Y)}$. Clearly, $d(Y) = 1$, we obtain

$$(4.9) \quad u(Y) = d^\gamma u(Y) \leq M := \sup_{Q_1(Y)} d^\gamma u.$$

By our construction, $d^\gamma u$ vanishes on $\partial_p Q_1$, so $\exists X_0 \in \overline{Q_1(Y)} \setminus \partial_p Q_1$ such that

$$(4.10) \quad M = d^\gamma u(X_0).$$

Let $r_0 := \frac{1}{2}d(X_0)$, we consider the intermediate region $Q_{r_0}(X_0)$. In this region, we have

$$\forall X \in Q_{r_0}(X_0), \quad d(X) \geq r_0.$$

Therefore, we conclude that

$$(4.11) \quad \sup_{Q_{r_0}(X_0)} u \leq r_0^{-\gamma} \sup_{Q_{r_0}(X_0)} d^\gamma u \leq r_0^{-\gamma} M \leq 2^\gamma u(X_0).$$

Now, we define $v = u - \frac{1}{2}u(X_0)$, then

$$v(X_0) = \frac{1}{2}u(X_0) \geq 2^{-1-\gamma} \sup_{Q_{r_0}(X_0)} u > 2^{-1-\gamma} \sup_{Q_{r_0}(X_0)} v.$$

From the first growth theorem, Theorem 12, we know $\exists \mu(n, \nu, \gamma, S) \in (0, 1]$ such that Theorem 12 holds with $\beta_1 = 2^{-1-\gamma}$. Now the above inequality tells us that v does not satisfy the measure condition in the first growth theorem. So

$$(4.12) \quad |\{v > 0\} \cap Q_{r_0}(X_0)| = \left| \left\{ u > \frac{1}{2}u(X_0) \right\} \cap Q_{r_0}(X_0) \right| > \mu |Q_{r_0}(X_0)|.$$

Now, we can show an integral estimate which is equivalent to the first growth theorem.

Theorem 14. *Let a function $u \in C^{2,1}(Q_r)$, where $Q_r := Q_r(Y)$, $Y = (y, s) \in \mathbb{R}^{n+1}$, $r > 0$. If $-u_t + Lu \geq 0$ in Q_r , then for arbitrary $p > 0$, we obtain*

$$(4.13) \quad u_+^p(Y) \leq \frac{N}{|Q_r|} \int_{Q_r} u_+^p dX,$$

where N only depends on n, ν, S, p .

Proof. Since the quantities we are considering are scaling invariant, we might assume $r = 1$. By the similar argument as above, we choose

$$(4.14) \quad \gamma = \frac{n+2}{p}.$$

We get

$$(4.15) \quad \left| \left\{ u > \frac{1}{2}u(X_0) \right\} \cap Q_{r_0}(X_0) \right| > \mu |Q_{r_0}(X_0)|,$$

where $\mu = \mu(n, \nu, \gamma, S) \in (0, 1]$. With the same notations as above we have

$$(4.16) \quad \begin{aligned} u_+^p(Y) &\leq M^p = (u)^p(X_0) = (2r_0)^{\gamma p} u^p(X_0) \\ &\leq \frac{(2r_0)^{\gamma p}}{\left| \left\{ u > \frac{1}{2}u(X_0) \right\} \cap Q_{r_0}(X_0) \right|} \int_{\left\{ u > \frac{1}{2}u(X_0) \right\} \cap Q_{r_0}(X_0)} (2u)^p dX \\ &\leq \frac{2^{\gamma p + p} r_0^{\gamma p}}{\mu |Q_{r_0}(X_0)|} \int_{Q_1} u_+^p dX. \end{aligned}$$

By our construction, it gives $r_0^{\gamma p} = r_0^{n+2}$, so the above estimate implies

$$(4.17) \quad u_+^p(Y) \leq \frac{N}{|Q_1|} \int_{Q_1} u_+^p dX.$$

□

We have seen the first growth theorem implies Theorem 14. Actually, we can also obtain the first growth theorem from Theorem 14. Indeed, from Theorem 14, we have

$$(4.18) \quad u_+^p(Y) \leq \frac{N}{|Q_r|} \int_{Q_r} u_+^p dX \leq N \frac{|\{u > 0\} \cap Q_r(Y)|}{|Q_r|} \sup_{Q_r} u_+^p \leq \mu N \sup_{Q_r} u_+^p.$$

Now it is trivial to see $\beta_1 \rightarrow 0^+$ as $\mu \rightarrow 0^+$ which is in consistent with the conditions in the first growth theorem.

Remark. The idea we used to find a non-degenerate point above will be also helpful when we prove the interior Harnack inequality.

5. SECOND GROWTH THEOREM

Before we establish the second growth theorem, Theorem 18, we need to prove some intermediate results based on the comparison principle and the Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate. Let us first do some preliminary calculations in order to carry out some comparison arguments.

For fixed number $\alpha > 0$ and $0 < \epsilon < 1$, in the cylinder $Q = B_r(0) \times (-r^2, (\alpha - 1)r^2)$, we can define

$$(5.1) \quad \psi_0 = \frac{(1 - \epsilon^2)(t + r^2)}{\alpha} + \epsilon^2 r^2$$

and

$$(5.2) \quad \psi_1 = (\psi_0 - |x|^2)_+$$

where $(\cdot)_+$ means positive part of the function. And we also define

$$(5.3) \quad \psi = \psi_1^2 \psi_0^{-q}$$

for some number $q \geq 2$ to be determined later. First of all, we notice ψ is $C^{2,1}$ in $\tilde{Q} := \{(x, t) \mid |x|^2 < \psi_0, -r^2 < t < (\alpha - 1)r^2\}$. It is clear that $-\psi_t + L\psi = 0$ if $\psi_0 \leq |x|^2$. Now if $\psi_0 > |x|^2$, by some computations, we obtain

$$-\psi_t + a_{ij}D_{ij}\psi = \psi_0^{-q} \left[8a_{ij}x_i x_j - 4\psi_1 \text{trace}(a_{ij}) + \frac{(1 - \epsilon^2)q}{\alpha\psi_0} \psi_1^2 - 2\frac{(1 - \epsilon^2)}{\alpha} \psi_1 \right].$$

Set $F_1 = \frac{2}{\alpha} + 8n\nu^{-1}$, and $\xi = \frac{\psi_1}{\psi_0}$ then

$$(5.4) \quad -\psi_t + a_{ij}D_{ij}\psi \geq \psi_0^{1-q} \left[\frac{(1 - \epsilon^2)q}{\alpha} \xi^2 - F_1 \xi + 8\lambda \right].$$

Pick

$$(5.5) \quad q = 2 + \frac{\alpha}{32(1 - \epsilon^2)},$$

so that the quadratic form in (5.4) is non-negative. Then we can conclude that

$$-\psi_t + \sum_{ij} a_{ij}D_{ij}\psi \geq 0, \quad \forall (x, t) \in Q$$

We also notice that

$$\psi(x, -r^2) \leq (\epsilon r)^{-2q+4}, \quad \forall |x| \leq r,$$

and

$$(5.6) \quad \psi(x, (\alpha - 1)r^2) \geq \frac{9}{16} r^{-2q+4}, \quad \forall |x| \leq \frac{r}{2}.$$

Finally, we notice that by the monotonicity of ψ with respect to $t \in [-r^2, (\alpha - 1)r^2]$ for $x = 0$, we obtain

$$(5.7) \quad \psi(0, t) \geq \frac{9}{16} r^{-2q+4}.$$

Now consider $-u_t + Lu \leq 0$, $u > 0$ in Ω . In Lemma 15, first of all, we establish that at least for a cylinder short enough, if we have a lower bound on some interior portion of the bottom, then it has a quantitative lower bound for the same portion on the top of the short cylinder. We also notice that the shortness only depends on the prescribed constants but not u . Then we can iterate this process to get a quantitative lower bound for an arbitrary time.

Lemma 15. *Let α be a positive constant and $-u_t + Lu \leq 0$, $u > 0$ in Ω . Suppose $Q := B_r(0) \times (-r^2, (\alpha - 1)r^2) \subset \Omega$ and $u > 0$ in $B_r(0) \times (-r^2, (\alpha - 1)r^2)$. Then for $\epsilon < \frac{1}{2}$, there are positive constants $C_1 = C_1(n, \nu)$ and $m = m(n, \nu, \alpha)$ such that if*

$$(5.8) \quad u \geq \ell$$

on $B_{\epsilon r}(0) \times \{-r^2\}$, then

$$(5.9) \quad u \geq C_1 \epsilon^m \ell$$

on $B_{\epsilon r}(0) \times \{(\alpha - 1)r^2\}$.

Proof. STEP 1:

$$(5.10) \quad -\psi_t + L\psi \geq b_i D_i \psi = -4\psi_1 \psi_0^{-q}(b, x) \geq -4|b|r\left(\frac{1}{2}\right)^{-2q} r^{2-2q}.$$

Consider

$$(5.11) \quad v = u - \ell(\epsilon r)^{2q-4} \psi$$

It is clear $\psi = 0$ for $|x| = r$. So we can conclude that $v \geq 0$ on $\partial_p \tilde{Q}$ by the above calculations. Finally, we apply the Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate to $-v$, we get

$$(5.12) \quad v \geq -N(n, \nu, S) \ell \epsilon^{-4} S$$

in \tilde{Q} . In other words, we get

$$(5.13) \quad u \geq \ell(\epsilon r)^{2q-4} \psi - N(n, \nu, S) \ell \epsilon^{-4} S.$$

So we have

$$(5.14) \quad u(x, (\alpha - 1)r^2) \geq \ell \epsilon^{-4} \left[\frac{9}{16}(\epsilon)^{2q} - N(n, \nu, S) S \right].$$

In particular, with the monotonicity of ψ with respect to t when $x = 0$, we obtain

$$(5.15) \quad u(0, t) \geq \ell \epsilon^{-4} \left[\frac{9}{16}(\epsilon)^{2q} - N(n, \nu, S) S \right], \quad \forall t \in [-r^2, (\alpha - 1)r^2].$$

By the similar calculations as above but with the variant of Aleksandrov-Bakelman-Pucci-Krylov-Tso applied to the region $B_r(0) \times (-r^2, (h\alpha - 1)r^2)$, we get

$$(5.16) \quad u(0, (h\alpha - 1)r^2) \geq \ell \epsilon^{-4} \left[\frac{9}{16}(\epsilon)^{2q} - N(n, \nu, S) S^{\frac{1}{n+1}} h^{\frac{1}{p} - \frac{1}{n+1}} \right].$$

Pick $h = h(n, \nu, S, \epsilon)$ small, we know that

$$(5.17) \quad \left[\frac{9}{16}(\epsilon)^{2q} - N(n, \nu, S) S^{\frac{1}{n+1}} (\alpha h)^{\frac{1}{p} - \frac{1}{n+1}} \right] \geq \frac{1}{2} \epsilon^{2q}.$$

So we can conclude

$$(5.18) \quad u(0, t) \geq C_1 \epsilon^k \ell$$

for $t \leq (h-1)r^2$ where $k = 2q-4$ and C_1 does not depend on u .

STEP 2: For arbitrary $t = (\alpha h - 1)r^2$, and $x \in B_\epsilon(0)$, we can use a slanted cylinder with radius ϵ to connect $B_\epsilon(x) \times \{(\alpha h - 1)r^2\}$ and $B_\epsilon(0) \times \{-r^2\}$. We can use a change of coordinate to reduce the slanted cylinder to a regular cylinder. We notice that with $k_i := \frac{y_i}{s}$ and $\frac{|y_i|}{s} = |k_i| \leq \frac{|y|}{s} \leq K$, then define $w_i = x_i - k_i t$ and $z = t$. In this coordinate, the slanted cylinder is transformed to a standard cylinder. The equation with respect to the new coordinate is

$$(5.19) \quad -u_z + \sum_{ij} a_{ij} D_{w_i w_j} u + \sum_i (b_i + k_i) D_{w_i} u \leq 0.$$

Then we apply the standard cylinder results to the equation with respect to coordinate (w, z) . We can do the same argument for all $x \in B_\epsilon(0)$. We have find h_x such that

$$(5.20) \quad u(x, (\alpha h_x - 1)r^2) \geq C_2 \epsilon^k \ell.$$

Since K is uniformly bounded above, so indeed, h_x and C_2 only depend on ϵ, n, ν, S . In particular, h_x can be uniformly bounded from below. Finally, we take

$$(5.21) \quad h_0 = \inf_{x \in B_\epsilon(0)} h_x > 0,$$

then for $x \in B_\epsilon(0)$, $t = (\alpha h_0 - 1)r^2$, we obtain

$$(5.22) \quad u(x, t) \geq C_2 \epsilon^k \ell.$$

□

STEP 3: Now for the general case, let α be positive constant as above, we can pick h_0 based on our discussion above. Finally by a simple iteration argument, we get the above result. Therefore $\exists m = m(n, S, \nu, \alpha)$, such that at least we conclude

$$(5.23) \quad u \geq C_1 \epsilon^m \ell$$

on $B_\epsilon(0) \times \{(\alpha - 1)r^2\}$.

Remark 16. In fact, in consistent with the Lemma 7.39 in [GL2], we can show that

$$u(x, (\alpha - 1)r^2) \geq C_3 \epsilon^m \ell$$

on $B_{\frac{1}{2}r}(0) \times \{(\alpha - 1)r^2\}$.

For a fixed point $Y = (y, s) \in \mathbb{R}^{n+1}$ with $s > 0$, and $r > 0$, we define the slanted cylinder

$$(5.24) \quad V_r = V_r(Y) := \left\{ X = (x, t) \in \mathbb{R}^{n+1}; \left| x - \frac{t}{s} y \right| < r, 0 < t < s \right\}.$$

Now the useful slanted cylinder lemma [FS] follows easily from Lemma 15 after we apply Lemma 15 to $1 - u$ after we multiply u by a constant to reduce our problem to the case $1 = \sup_{V_r(Y)} u$.

Lemma 17 (Slanted Cylinder Lemma). *Let a function $u \in C^{2,1}(\overline{V_r})$ satisfy $-u_t + Lu \geq 0$ in a slanted cylinder V_r , which is defined in (5.24) with $Y = (y, s) \in \mathbb{R}^{n+1}$, $s > 0$, $r > 0$, such that*

$$(5.25) \quad K^{-1}r|y| \leq s \leq Kr^2$$

where $K > 1$ is a constant. In addition, suppose $u \leq 0$ on $D_r := B_r(0) \times \{0\}$. Then

$$(5.26) \quad u(Y) \leq \beta_2 \sup_{V_r(Y)} u_+$$

with a constant $\beta_2 = \beta_2(\nu, n, K, S) < 1$.

With the slanted cylinder lemma, we are ready to prove the second growth theorem. The slanted cylinder lemma, i.e., Lemma 17 above plays a crucial role in this section to build a connection between different time slides. The second growth theorem helps us construct some control of the oscillation between different time slides. We follow the arguments in [FS].

Theorem 18 (Second Growth Theorem). Let a function $u \in C^{2,1}(\overline{Q_r})$, where $Q_r := Q_r(Y)$, $Y = (y, s) \in \mathbb{R}^{n+1}$, $r > 0$, and let $-u_t + Lu \geq 0$ in Q_r . In addition, suppose $u \leq 0$ on $D_\rho := B_\rho(z) \times \{\tau\}$, where $B_\rho(z) \subset B_r(y)$ and

$$(5.27) \quad s - r^2 \leq \tau \leq s - \frac{1}{4}r^2 - \rho^2.$$

Then

$$(5.28) \quad u(Y) \leq \beta_3 \sup_{Q_r(Y)} u_+$$

where $\beta_3 := \beta_3(n, \nu, \rho/r, S) < 1$ is a constant.

Proof. After rescaling and translation in \mathbb{R}^{n+1} , we reduce our problem to $r = 1$, and $(z, \tau) = (0, 0) \in \mathbb{R}^{n+1}$. For an arbitrary point $Y' \in Q_{\frac{1}{2}}(Y)$, we can apply the slanted cylinder lemma to the slanted cylinder $V_\rho(Y') \subset Q_1(Y)$. Note that in this situation, the constant K in slanted cylinder lemma only depends on ρ . Therefore, with the parameter β_2 from the slanted cylinder lemma, we have

$$u(Y') \leq \beta_2 \sup_{V_\rho(Y')} u_+ \leq \beta_2 \sup_{Q_1(Y)} u_+.$$

The above estimate holds for all $Y' \in Q_{\frac{1}{2}}(Y)$. Therefore we get

$$\sup_{Q_{\frac{1}{2}}(Y)} u_+ \leq \beta_2 \sup_{Q_1(Y)} u_+.$$

□

Now we establish an estimate similar to above with more explicit dependence of the constant on the ratio ρ/r .

Lemma 19. Let a function $v \in C^{2,1}(\overline{Q_r})$ satisfy $v \geq 0$, $-v_t + Lv \leq 0$ in $Q_r := Q_r(Y)$, $Y = (y, s) \in \mathbb{R}^{n+1}$, $r > 0$. For arbitrary disks $D_\rho := B_\rho(z) \times \{\tau\}$ and $D^0 := B_{\frac{r}{2}}(y) \times \{\sigma\}$, such that $B_\rho(z) \subset B_r(y)$ and

$$(5.29) \quad s - r^2 \leq \tau < \tau + h^2 r^2 \leq \sigma \leq s,$$

where $h \in (0, 1)$ is a constant. Then

$$(5.30) \quad \inf_{D_\rho} v \leq \left(\frac{2r}{\rho} \right)^\gamma \inf_{D^0} v$$

where $\gamma = \gamma(n, \nu, h, S)$.

Proof. Without loss of generality, we may assume $m := \inf_{D_\rho} v > 0$, $r = 1$, $z = 0$, $\tau = 0$, $\sigma = s = h^2$. So $D_\rho = B_\rho(0) \times \{0\}$. We can apply an additional linear transformation along t -axis, we can also reduce the proof to the case $h = 1$. Now fix the integer k such that $2^{-k-1} < \rho \leq 2^{-k}$, and for $j = 0, 1, \dots$, and define $y^j := y^* + 2^{-j}(y - y^*)$, $B^j := B_{2^{-j}}(y^j)$, where $y^* := \frac{\rho}{1-\rho}y$, $Y^j := (y^j, 4^{-j})$, $Q^j := Q_{2^{-j}}(Y^j)$, $D^j := B_{2^{-j-1}}(y^j) \times \{4^{-j}\}$. By construction, $0 = y^* + \rho(y - y^*)$, so that

$$B_\rho(0), B^j \in \{B_\theta(y^* + \theta(y - y^*)); 0 \leq \theta \leq 1\}.$$

Then by the assumption, $B_\rho(0) \subset B_1(y)$ it follows $|y| \leq 1 - \rho$, $|y - y^*| \leq 1$, and

$$B^{k+1} \subset B_\rho(0) \subset B^k \subset B^{k-1} \subset \dots \subset B^1 \subset B^0 = B_1(y).$$

Now apply Theorem 18 to the function $u = 1 - \frac{1}{m}v$ in Q^k with

$$r = 2^{-k}, \rho = 2^{-k-1}, Y = Y^k, z = 0, \tau = 0.$$

Then we conclude that

$$\sup_{D^k} u \leq \sup_{Q_{2^{-k-1}}(Y^k)} u \leq \beta_3 \sup_{Q^k} u \leq \beta_3 = \beta_2(n, \nu, S, \frac{1}{2}) < 1,$$

which is equivalent to

$$\inf_{D_\rho} v = m \leq (1 - \beta_3)^{-1} \inf_{D^k} v = 2^\gamma \inf_{D^k} v,$$

where $\gamma := -\log_2(1 - \beta_3) > 0$. Similarly, if $k \geq 1$, we also have

$$\inf_{D^j} v \leq 2^\gamma \inf_{D^{j-1}} v,$$

for $j = 1, 2, \dots, k$. Finally we have

$$\inf_{D_\rho} v \leq 2^\gamma \inf_{D^k} v \leq 2^{2^\gamma} \inf_{D^{k-1}} v \leq \dots \leq 2^{(k+1)\gamma} \inf_{D^0} v \leq \left(\frac{2r}{\rho}\right)^\gamma \inf_{D^0} v.$$

□

6. INTERIOR HARNACK INEQUALITY

We also need the third growth theorem in order to establish the interior Harnack inequality. The first growth theorem tells us if $\mu \rightarrow 0^+$ then $\beta_1 \rightarrow 0^+$. The third growth theorem tells us if we have a nice control of the measure of the set $\{u > 0\}$ near the bottom, then we can have a more precise estimate. In other words, if we have the similar measure condition for

$$Q^0 := Q_{\frac{r}{2}}(Y^0), Y^0 = \left(y, s - \frac{3}{4}r^2\right).$$

Then if $\mu < 1$, then $\beta_1 < 1$. The proof of it is long and technical but independent of the specific structure of the equations. One can find a detailed proof in, for example, [FS, KS, GC]. We just formulate the results here.

Theorem 20 (Third Growth Theorem). *Let a function $u \in C^{2,1}(\overline{Q_r})$, where $Q_r = Q_r(Y)$, $Y = (y, s) \in \mathbb{R}^{n+1}$, $r > 0$, and let $-u_t + Lu \geq 0$ in Q_r . In addition, we assume*

$$(6.1) \quad |\{u > 0\} \cap Q^0| \leq \mu |Q^0|,$$

where

$$(6.2) \quad Q^0 := Q_{\frac{r}{2}}(Y^0), \quad Y^0 = \left(y, s - \frac{3}{4}r^2\right)$$

and $\mu < 1$ is a constant. Then we have

$$(6.3) \quad \mathcal{M}_{\frac{r}{2}}(Y) \leq \beta \mathcal{M}_r(Y)$$

with a constant

$$\beta := \beta(n, \nu, S, \mu) < 1.$$

Corollary 21. *Let a function $v \in C^{2,1}(\overline{Q_r})$ be such that $v \geq 0$, $-v_t + Lv \leq 0$ in Q_r , and*

$$(6.4) \quad |\{v \geq 1\} \cap Q^0| > (1 - \mu) |Q^0|.$$

Then

$$(6.5) \quad v \geq 1 - \beta > 0$$

on $Q_{\frac{r}{2}}$ where $\beta = \beta(n, \nu, \mu, S) < 1$ for $\mu < 1$.

With the above preparation, we now are ready to establish the interior Harnack inequality.

Theorem (Interior Harnack Inequality). *Suppose $u \in C^{2,1}(Q_{2r}(Y)) \cap C(\overline{Q_{2r}(Y)})$ and $-u_t + Lu = 0$ in $Q_{2r}(Y)$, $Y = (y, s) \in \mathbb{R}^{n+1}$ and $r > 0$. If $u \geq 0$, then*

$$(6.6) \quad \sup_{Q^0} u \leq N \inf_{Q_r} u,$$

where $N = N(n, \nu, S)$ and $Q^0 = B_r(y) \times (s - 3r^2, s - 2r^2)$.

We will build a non-degenerate intermediate region to get a quantitative relation between two regions we are interested in with the help of three growth theorems.

Proof. After rescaling and translating as necessary, we can assume $Y = 0$ and $r = 1$. Now $Q_1 = B_1(0) \times (-1, 0)$, $Q^0 = B_1(0) \times (-3, -2)$. It is easy to see that if we define $d(X) := \sup\{\rho > 0 : Q_\rho(X) \subset Q_2(0)\}$, then $d(X) \geq 1$ in Q^0 . Hence, if we consider $Q^1 := B_2(0) \times (-3, -2)$ we conclude that

$$(6.7) \quad \sup_{Q^0} u \leq M := \sup_{Q^1} d^\gamma u,$$

where γ is chosen at the same as the γ in Lemma 19 with $h = \frac{1}{2}$. From the discussion before Theorem 14, we can find $\exists X_0 \in \overline{Q^1} \setminus [\partial_p Q^1 \cap \partial_p Q_2]$ such that

$$(6.8) \quad d^\gamma u(X_0) = M.$$

Similarly as above, we define

$$(6.9) \quad \rho = \frac{1}{4}d(X_0) \in (0, \frac{1}{2}],$$

and

$$(6.10) \quad Q_0 = Q_\rho(X_0) \cap \left\{u > \frac{1}{2}u(X_0)\right\}.$$

By the above discussion, we obtain

$$|Q_0| > \mu_1 |Q_\rho(X_0)|$$

for some constant $\mu_1 = \mu_1(n, \nu, S, \gamma) > 0$. Now we apply the Corollary 21 with

$$v = \frac{2}{u(X_0)}u, \quad Q_r = Q_{2\rho}(Y_0), \quad Y_0 = (x_0, t_0 + 3\rho^2), \quad Q^0 = Q_\rho(X_0), \quad 1 - \mu = \mu_1.$$

Then we have

$$(6.11) \quad u \geq \beta u(X_0)$$

on $Q_\rho(Y_0)$ with $\beta = \beta(n, \nu, S) > 0$. Next we apply Lemma 19 with

$$v = u, \quad r = 2, \quad D_\rho = B_\rho(x_0) \times \{t_0 + 2\rho^2\} \subset \overline{Q_\rho(Y_0)},$$

and

$$D^0 = B_1(0) \times \{\tau\}, \quad \forall \tau \in (-1, 0).$$

So we have

$$(6.12) \quad \beta u(X_0) \leq \inf_{D_\rho} u \leq \left(\frac{4}{\rho}\right)^\gamma \inf_{Q_1(0)} u.$$

Finally, with the help of the intermediate region, we conclude that

$$(6.13) \quad \sup_{Q^0} u \leq M = d^\gamma u(X_0) = (4\rho)^\gamma u(X_0) \leq \beta^{-1} 4^{2\gamma} \inf_{Q_1(0)} u.$$

Taking $N = N(n, \nu, S) = \beta^{-1} 4^{2\gamma}$ gives the desired result. \square

It is well-known that it is easy to derive the Hölder continuity of solutions from the Harnack inequality by standard oscillation and iteration arguments.

Theorem 22. *Suppose $u \in W_p$ where $p < n + 1$ is from Theorem 2, and u is a solution of $-u_t + Lu = 0$ in Q_r . Then u is Hölder continuous in $Q_{\frac{r}{2}}$.*

7. APPROXIMATION

In all the proofs from above sections, we always assume u is $C^{2,1}$ in stead of W_p where $p < n + 1$ is from Theorem 2. In this section, we briefly show we can use an approximation argument to show that all results hold for $u \in W_p(Q_{2r})$ in the sense of Definition 1 and p is from the variant Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate (1.8). Throughout, we assume

$$(7.1) \quad u \geq 0, \quad -u_t + Lu = -u_t + \sum_{ij} a_{ij} D_{ij} u + \sum_i b_i D_i u = 0$$

in Q_{2r} . We can approximate a_{ij} , b_i and u by smooth functions $a_{ij}^\epsilon \rightarrow a_{ij}$, $b_i^\epsilon \rightarrow b_i$ a.e. as $\epsilon \rightarrow 0^+$. And $u^\epsilon \rightarrow u$ in $W_p^{2,1}$ as $\epsilon \rightarrow 0^+$. Then

$$(7.2) \quad f^\epsilon = -u_t^\epsilon + L^\epsilon u^\epsilon = -u_t^\epsilon + \sum_{ij} a_{ij}^\epsilon D_{ij} u^\epsilon + \sum_i b_i^\epsilon D_i u^\epsilon \rightarrow 0$$

in $L_{loc}^p(Q_{2r})$ as $\epsilon \rightarrow 0^+$. We know the existence of solutions to equations with smooth coefficients, therefore we can write

$$u^\epsilon = v^\epsilon + w^\epsilon,$$

where

$$-v_t^\epsilon + L^\epsilon v^\epsilon = 0$$

in Q_{2r} and

$$v^\epsilon = u^\epsilon$$

on $\partial_p Q_{2r}$:

$$\begin{aligned} -w_t^\epsilon + L^\epsilon w^\epsilon &= f^\epsilon, \\ w^\epsilon &= 0 \end{aligned}$$

on $\partial_p Q_{2r}$. By the variant Aleksandrov-Bakelman-Pucci-Krylov-Tso estimate (1.8), we know $w^\epsilon \rightarrow 0$ in L^∞ and v^ϵ satisfies the Harnack inequality. Finally, by an easy limiting argument, u also satisfies the Harnack inequality.

8. APPENDIX

As we mentioned in the introduction, when drift $b \in L_{x,t}^{n+1}$, we do not expect the solution to have Hölder continuity since $L_{x,t}^{n+1}$ is supercritical with respect to the parabolic scaling. In this appendix, we present a concrete example. We consider the parabolic equation in $1+1$ dimensions with drift $b \in L_{x,t}^2(\mathbb{R}^2)$,

$$(8.1) \quad u_t + b \nabla u - \Delta u = 0.$$

We define for $t \in [0, 1]$

$$(8.2) \quad b(x, t) = a(t) \begin{cases} 1 & -r(t) \leq x < 0 \\ -1 & 0 < x \leq r(t) \\ 0 & x \notin [-r(t), 0) \cup (0, r(t)] \end{cases}$$

and if $t \notin [0, 1]$, $b = 0$. We set $a(t) = (1-t)^{-\beta}$ and $r(t) = (1-t)^\alpha$ where β and α to be determined later. First of all, by the integrability condition of b , we see $\int_0^1 (1-t)^{\alpha-2\beta} < \infty$, we get $\alpha - 2\beta > -1$.

We try to construct an odd function ϕ so that we can do a comparison argument. For $0 \leq x \leq 1$, we define $\phi(x) = \sin(\pi x/2)$ and $\phi = 1$ for $x > 1$. Notice that for $x \in [0, 1]$, we have $-\Delta \phi \leq C \phi$ for some constant C . In particular, based on our specific choice, we take the constant $C = (\frac{2}{\pi})^2$. Finally, we extend this ϕ oddly to the whole line. We consider

$$(8.3) \quad v(x, t) = \exp \left[-C \int_0^t (1-s)^{-2\alpha} \right] \phi(x/r(t))$$

which requires $-2\alpha > -1$. We try to verify on $(0, r(t))$, v is a subsolution for $t \in [0, 1]$

$$(8.4) \quad v_t = \exp \left[-C \int_0^t (1-s)^{-2\alpha} \right] (x\alpha(1-t)^{-\alpha-1}\phi' - C(1-t)^{-2\alpha}\phi)$$

$$(8.5) \quad b \nabla v = -(1-t)^{-\beta}(1-t)^{-\alpha}\phi' \exp \left[-C \int_0^t (1-s)^{-2\alpha} \right]$$

$$(8.6) \quad -\Delta v = -(1-t)^{-2\alpha}\phi'' \exp \left[-C \int_0^t (1-s)^{-2\alpha} \right]$$

By construction,

$$(8.7) \quad -(1-t)^{-2\alpha}\phi'' \exp \left[-C \int_0^t (1-s)^{-2\alpha} \right] - C(1-t)^{-2\alpha}\phi \exp \left[-C \int_0^t (1-s)^{-2\alpha} \right] \leq 0.$$

We only need to verify that

$$(8.8) \quad C(x\alpha(1-t)^{-a-1}\phi') - (1-t)^{-\beta}(1-t)^{-\alpha}\phi' \exp\left[-C\int_0^t(1-s)^{-2\alpha}\right] \leq 0$$

for $x \in (0, r(t))$ and $t \in [0, 1)$. Since ϕ' is nonnegative, it suffices to check

$$(8.9) \quad x\alpha(1-t)^{-a-1} - (1-t)^{-\beta}(1-t)^{-\alpha} \leq 0, \quad x \in [0, r(t)), \quad t \in [0, 1).$$

$$(8.10) \quad x\alpha(1-t)^{-a-1} \leq \alpha(1-t)^{-\alpha-1}(1-t)^\alpha = \alpha(1-t)^{-1}.$$

We pick $-\beta - \alpha + 1 < 0$. Also $-2\alpha > -1$ and $\alpha - 2\beta > -1$. So $\alpha \in (\frac{1}{3}, \frac{1}{2})$, we can pick $\alpha = \frac{5}{12}$. Then we pick $\beta = \frac{2}{3}$. This pair satisfies all of our conditions. So with our choice of α and β , we can see our $v(x, t)$ is a subsolution to our equation in $[0, r(t)) \times [0, 1)$.

By the symmetry of our equation and the oddness of ϕ , v is a subsolution of our equation $[0, r(t)) \times [0, 1)$ and is a supersolution on $(-r(t), 0] \times [0, 1)$. If we take u to be the solution of our equation with initial data $v(x, 0)$. Then when t approach 1, we look at the oscillation on the ball $B_{r(t)}(0)$. We have

$$(8.11) \quad 2 \leq Osc_{B_{r(t)}(0)} v \leq Osc_{B_{r(t)}(0)} u.$$

Since $r(t) \rightarrow 0$ as $t \rightarrow 1$, we conclude that u will have a discontinuity at the origin when $t \rightarrow 1$.

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